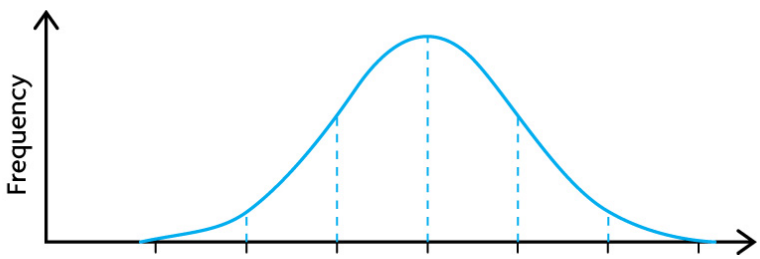
Section 5.5 Z-Scores

Example 1

Alexis plays in her school jazz band. Band members practice an average of 16.5 h per week, with a standard deviation of 4.2 h. Alexis practices an average of 22 h per week.

Set up a normal distribution curve, to help estimate the percent of the band that, on average, practices a greater number of hours than Alexis.



But this is just an estimate! How can we find an exact answer?

Recall: Facts about Normal Distribution Curves

- Each normal distribution curve has its own mean, μ, and standard deviation, σ.
- Because different populations have different means and standard deviations, their curves will not be exactly the same but all normal distribution curves are bell-shaped.
- To compare different normal distribution curves we must standardize the normal distribution. This requires using Z-Scores!

The Z-Score

- A standardized value that indicates the number of standard deviations of a data value above or below the mean.

- The greater the numerical value of the z-score, the farther it is from the mean.

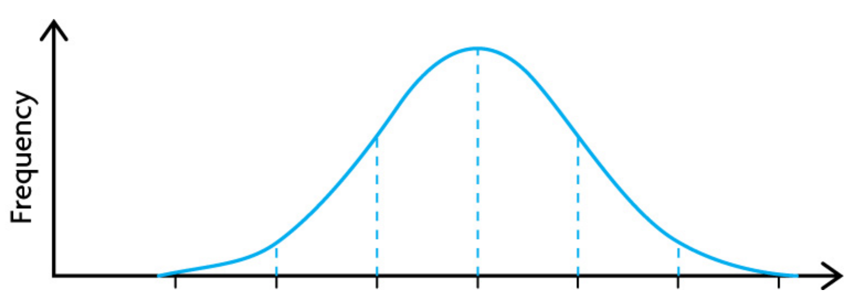
- To determine the z-score we use the formula:

$$z = \frac{x - \mu}{\sigma}$$

Then refer to the chart p.580-581 at the back of your book. The z-score will give you a percent for the area under the curve, less than or equal to the data value.

Let's redo Alexis's problem. Find the EXACT percent of the band that practices a greater number of hours than Alexis.

$$z = \frac{x - \mu}{\sigma}$$

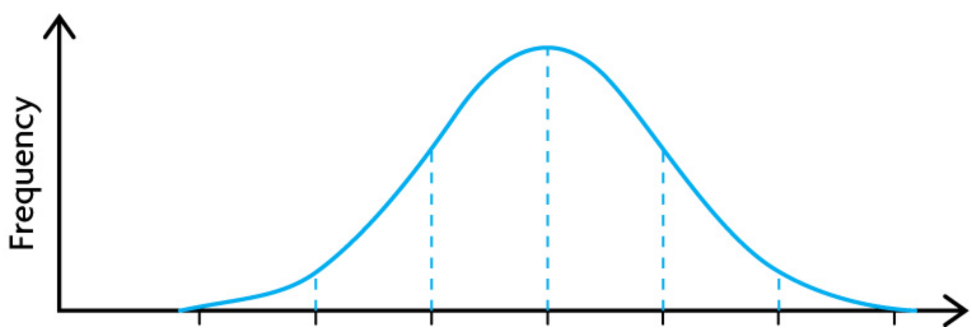


Note

Notice the z-score table goes from 2.99 to -2.99. That's because the normal distribution has been standardized!

It has a mean, $\mu = 0$, and a standard deviation, $\sigma = 1$.

The Standard Normal Distribution Curve:



The purpose of the z-score is to determine the number of standard deviations a data value is from the mean.

- A positive z-score means the value is to the right or above the mean.
- A negative z-score means the value is to the left or below the mean.
- The total area under the standard normal distribution is 1.

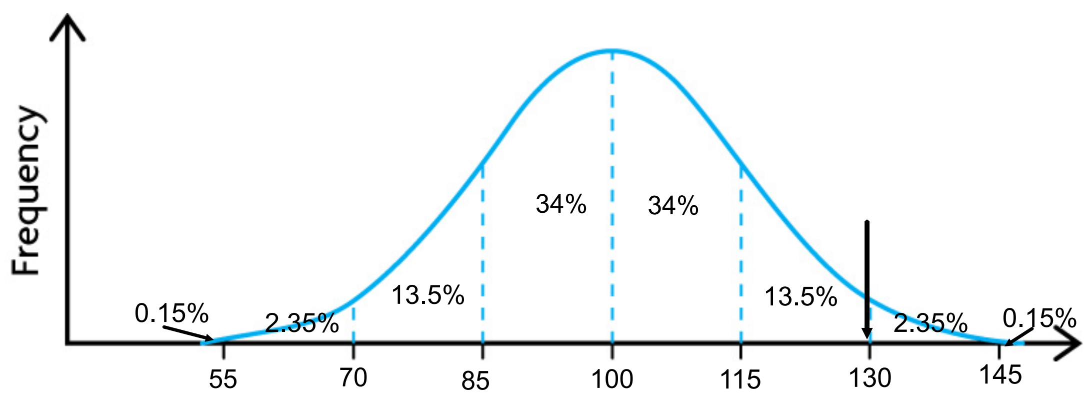
We standardize the normal curve to:

- talk about particular scores within a set of data
- tell other people about whether or not a score is above or below average
- indicate how far away a particular score is from the average
- compare scores from different sets of data and figure out which score is better

Example 1:

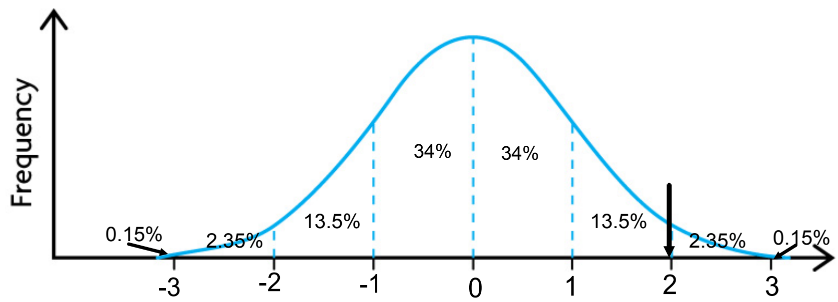
IQ tests are normally distributed with a mean of 100 and a standard deviation of 15.

a) Draw the normal distribution curve, labeling the mean and standard deviation. What percentage of students achieved less than the 130 mark?



There would be 97.5% of students achieving less than 130.

b) Draw the standard normal distribution curve and indicate where the 130 mark is found.



+ z-scores
- z-scores

c) Using the z-score formula and then the z-score table (pg 580-581) check what percentage of students achieved less than the 130 mark?
Was there any difference in your answers from (b) and (c)? Explain.

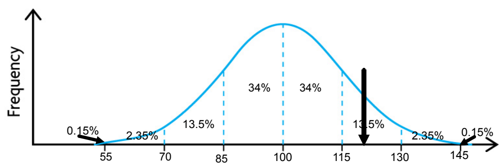
$$z = \frac{130 - 100}{15}$$

$$= 2$$

0.9772 or 97.72%

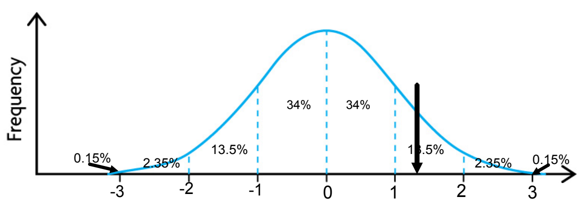
Yes - my first calculation was 97.5% and using z-score I got 97.7%. The error can be attributed to rounding. The z-score uses more decimal places and is more accurate based on the normal distribution curve.

d) Using your diagram from (a), estimate the percentage of students who achieved less than 120.



50% + 34% + 6% = 90%

e) Using your diagram from (b), the z-score formula and the z-score table determine the percentage of students who achieved less than 120.



$$z = \frac{120 - 100}{15}$$

$$= 1.33$$

.9082 so about 91%

f) Was your estimate reasonable when you compared it to the z-score?

Yes! They were very close!!

g) Why is the z-score more reliable than estimating using standard deviation?

The z-score has less rounding error!

h) What percentage of students achieved more than 120?

If approximately 91% achieved less than 120, than 100% - 91% or 9% achieved more than 120.

http://www.ltcconline.net/greenl/courses/201/probdist/zScore.htm

Example 2:
Two students competed in a nation-wide mathematics competition and received these scores.

Alma 70 Bruce 80

If $\mu = 66$ and $\sigma = 10$, find their z-scores.

Alma:

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{70 - 66}{10}$$

$$= 0.4$$

Bruce:

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{80 - 66}{10}$$

$$= 1.4$$

Example 3:
On the math placement test at Memorial University of Newfoundland, the mean score was 62 and the standard deviation was 11. If Mark's z-score was 0.8, what was his actual exam mark?

$$z = \frac{x-\mu}{\sigma}$$

$$0.8 = \frac{x - 62}{11}$$

$$\frac{0.8}{1} = \frac{x - 62}{11}$$

cross multiply!!

$$x - 62 = 8.8$$
$$+ 62 \quad + 62$$

$$x = 70.8$$

He scored 70.8

Example 4:

On her first math test, Susan scored 70%. The mean class score was 65% with a standard deviation of 4%. On her second test she received 76%. The mean class score was 73% with a standard deviation of 10%.

a) Without performing any calculations, which test do you think she did better on?

b) By calculating 2 separate z-scores, which test did Susan perform better with respect to the rest of her class?

First test

$$z = \frac{70 - 65}{4}$$

$$= 1.25$$

She did better than 89.44% of the class.

Second test

$$z = \frac{76 - 73}{10}$$

$$= 0.30$$

She did better than 61.79% of the class.

With respect to the rest of her class Susan did better on the first test!

Example 6: Quality Control

Red candy hearts are packaged according to weight with a mean of 300 g and a standard deviation of 8 g. Packages with weights less than 290 g and more than 312 g are rejected by quality control workers.

a) If 50 000 packages are produced each day, how many packages would quality control expect to reject in a day?

minimum

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{290 - 300}{8}$$

$$= \frac{-10}{8}$$

$$= -1.25$$

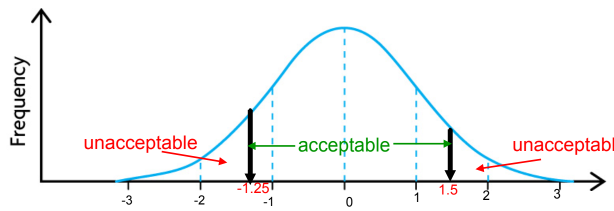
maximum

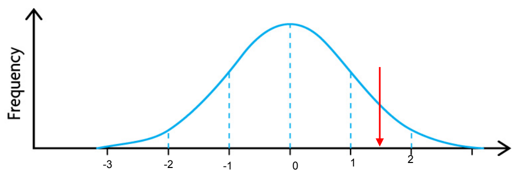
$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{312 - 300}{8}$$

$$= \frac{12}{8}$$

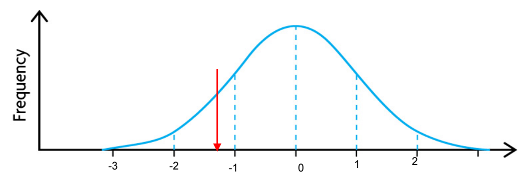
$$= 1.5$$





z = 1.5

93.32% will weigh less than 312 g.
100% - 93.32% or 6.68%
will weigh more than 312 g.



z = -1.25

10.56% will weigh less than 290 g.

so 6.68% + 10.56% or 17.24% are outside the limits.

If 50 000 packages are produced in a day:

17.24% of 50 000 are rejected
0.1724 x 50 000
8620

8 620 packages are rejected each day!

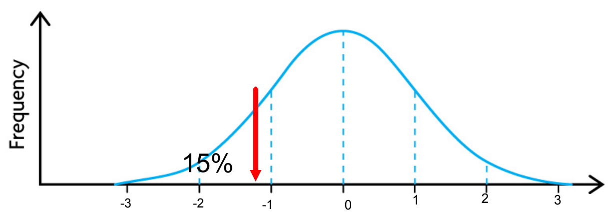
b) What advice would you give this company?

Adjustments to the packaging process MUST be made. 8620 packages are too many to reject!

Example 7: Warranties

Cars are undercoated as a protection against rust. A car dealer determines the mean life of protection is 65 months and the standard deviation is 4.5 months.

a) What guarantee should the dealer give so that fewer than 15% of the customers will return their cars?



the z-score for 15% is -1.035

In months, this represents $z = \frac{x-\mu}{\sigma}$

$$(-1.035) = \frac{x - 65}{4.5}$$

$$\frac{(-1.035)}{1} = \frac{x - 65}{4.5}$$

$$-4.6575 = x - 65$$
$$+ 65 \qquad + 65$$

$$60.3425 = x$$

The dealer should offer a 60 month warranty.

b) The dealer creates a fund, based on the guarantee, from which refunds and repairs are made. It is estimated that about 2500 cars will be undercoated annually. The average repair on returned cars is about $165. How much money should be placed in the fund to cover customer returns?

If 15% of cars will be returned before 60 months:

15% of 2500 cars is
0.15 x 2500
375 cars

375 cars at $165 each is $61 875 to place in the fund to cover returns.

c) What is the probability that an undercoated car, chosen at random, will be returned in 5 years?

5 years = 5 x 12 months = 60 months

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{60 - 65}{4.5}$$

$$= \frac{-5}{4.5}$$

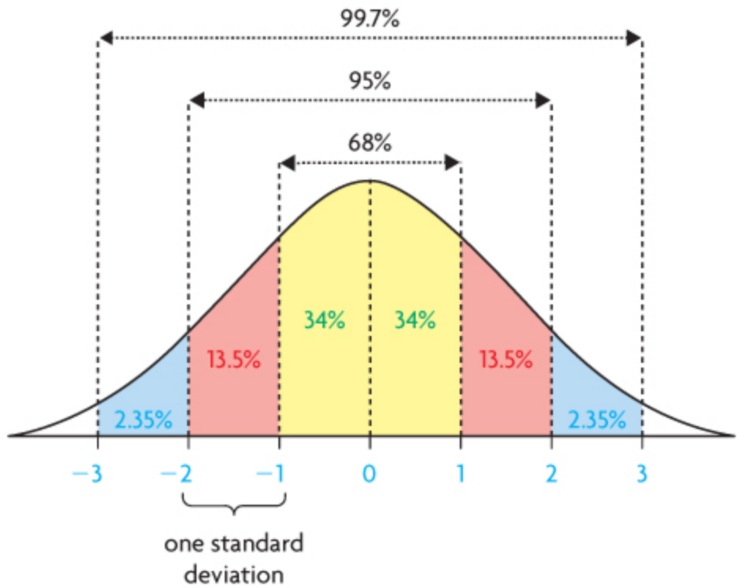
$$= -1.11$$

so there is a 13.35% chance that a car will return in 5 years.

In Summary

Key Ideas

- The standard normal distribution is a normal distribution with mean, μ, of 0 and a standard deviation, σ, of 1. The area under the curve of a normal distribution is 1.
- *Z*-scores can be used to compare data from different normally distributed sets by converting their distributions to the standard normal distribution.



Need to Know

- A *z*-score indicates the number of standard deviations that a data value lies from the mean. It is calculated using this formula:

$$z = \frac{x - \mu}{\sigma}$$

- A positive *z*-score indicates that the data value lies above the mean. A negative *z*-score indicates that the data value lies below the mean.
- The area under the standard normal curve, to the left of a particular *z*-score, can be found in a *z*-score table or determined using a graphing calculator.

Page 292 - 293

1ab, 2ab, 3a, 4, 6ab, 7ab, 8,

9,10,13, 15, 16,17,18

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986